Assignment 2

Exercise 2.1

1. Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.
2. Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = \log(1+x^2)$, $u_t(x,0) = 4+x$.
5. (*The hammer blow*) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c, a/c, 3a/2c, 2a/c$, and $5a/c$. [*Hint*: Calculate
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\{\text{length of } (x-ct, x+ct) \cap (-a,a)\}.$$
Then $u(x, a/2c) = (1/2c)\{\text{length of } (x - a/2, x + a/2) \cap (-a,a)\}$. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner fro each case.]
6. In Exercise 5, find the greatest displacement, $\max_x u(u,t)$, as a function of t.
7. If both ϕ and ψ are odd functions of x, show that the solution $u(x,t)$ of the wave equation is also odd in x for all t.
8. A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form
$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \qquad (\text{“spherical wave equation”}).$$
   (a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.
   (b) Solve for v using (3) and thereby solve the spherical wave equation.
   (c) Use (8) to solve it with initial conditions $u(r,0) = \phi(r), u_t(r,0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.

   (*Hint*: Factor the operator as we did for the wave equation.)
10. Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

Exercise 2.2

1. Use the energy conservation of the wave equation to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$.(*Hint*: Use the first vanishing theorem in Section A.1.)
2. For a solution $u(x,t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.
   (a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.
   (b) Show that both $e(x,t)$ and $p(x,t)$ also satisfy the wave equation.
3. Show that the wave equation has the following invariance properties.
   (a) Any translate $u(x-y,t)$, where y is fixed, is also a solution.
   (b) Any derivative, say u_x, of a solution is also a solution.
   (c) The dilated function $u(ax, at)$ is also a solution, for any constant a.

   identity
5. For the *damped* string, equation (1.3.3), show that the energy decreases.

Exercise 2.3

2. Consider a solution of the diffusionn equation $u_t = u_{xx}$ in $\{0 \le x \le l, 0 \le t < \infty\}$.

   (a) Let $M(T)$ = the maximum of $u(x,t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?

   (b) Let $m(T)$ = the minimum of $u(x,t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?

3. Consider the diffusion equation $u_t = u_{xx}$ in the interval $(0,1)$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 1-x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.

   (a) Show that $u(x,t) > 0$ at all interior points $0 < x < 1$, $0 < t < \infty$.

   (b) For each $t > 0$, let $\mu(t)$ = the maximum of $u(x,t)$ over $0 \le x \le 1$. Show that $\mu(t)$ is a decreasing (i.e., nonincreasing) function of t. (*Hint*: Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t),t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)

   (c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1, 0 < t < \infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.

   (a) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

   (b) Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$.

   (c) Use the energy method to show that $\int_0^1 u^2 dx$ is a strictly decreasing funciton of t.

5. The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.

   (a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2, 0 \le t \le 1\}$.

   (b) Where precisely does our proof of the maximum principle break down for this equation?

6. Prove the *comparison principle* for the diffusion equation: If u and v are two solutions, and if $u \le v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \le v$ for $0 \le t < \infty$, $0 \le x \le l$.

7. (a) More generally, if $u_t - ku_{xx} = f$, $v_t - kv_{xx} = g$, $f \le g$ and $u \le v$ at $x = 0$, $x = l$ and $t = 0$, prove that $u \le v$ for $0 \le x \le l, 0 \le t < \infty$.

   (b) If $v_t - v_{xx} \ge \sin x$ for $0 \le x \le \pi$, $0 < t < \infty$, and if $v(0,t) \ge 0$, $(\pi,t) \ge 0$ and $v(x,0) \ge \sin x$, use part (a) to show that $v(x,t) \ge (1 - e^{-t}\sin x)$.

Extra 1. Consider the diffusion equation $u_t = ku_{xx} + au$ in $(0 < x < 1, 0 < t < \infty)$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = sin(\pi x)$ where $k > 0, a \le 0$.

(1)Show that $0 < u(x,t) < 1, \forall t > 0, 0 < x < 1$.

(2)Show that $u(x,t) = u(1-x,t), \forall t \ge 0, 0 \le x \le 1$.

Extra 2. (a) Prove the following generalized Maximum Principle: if $u_t - ku_{xx} \le 0$ in $R = [0,l] \times [0,T]$,where $k > 0$,then

$$\max_R u(x,t) = \max_{\partial R} u(x,t)$$

(b)Show that if $v(x,t)$ satisfies

$$v_t = kv_{xx} + f(x,t), -\infty < x < +\infty, 0 < t < T$$
$$v(x,0) = 0$$

then$v(x,t) \leq T \max_{-\infty<x<+\infty, 0<t<T} f(x,t)$

Exercise 2.4

1. Solve the diffusion equation with the initial condition
$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$
Write your answer in terms of $\mathscr{E}rf(x)$.

2. Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.

5. Prove properties (a) to (e) of the diffusion equation (1).

8. Show taht for any fixed $\delta > 0$(no matter how small),
$$\max_{\delta \leq |x| < \infty} S(x,t) \to 0 \qquad \text{as } t \to 0.$$
[This means that thee tail of $S(x,t)$ is "uniformly small".]

11. (a) Consider the diffusion equation on the whole line with the usual initial condition $u(x,0) = \phi(x)$. If $\phi(x)$ is an *odd* function, show that the solution $u(x,t)$ is also an *odd* function of x. (*Hint*: Consider $u(-x,t) + u(x,t)$ and use the uniqueness.)
    (b) Show that the same is true if "odd" is replaced by "even".
    (c) Show that the analogous statements are true for the wave equation.

14. Let $\phi(x)$ be a continuous function such that $|\phi(x)| \leq Ce^{ax^2}$. Show that formula (8) for the solution of the diffusion equation makes sense for $0 < t < 1/(4ak)$, but not necessarily for larger t.

15. Prove the uniqueness of the diffusion problem with Neumann boundary conditions:
$$u_t - k_{xx} = f(x,t) \quad \text{for } 0 < x < l, t > 0 \quad u(x,0) = \phi(x)$$
$$u_x(0,t) = g(t) \quad u_x(l,t) = h(t)$$
by the energy method.

16. Solve the diffusion equation with constant dissipation:
$$u_t - ku_{xx} + bu = 0 \qquad \text{for} - \infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$
where $b > 0$ is a constant.(*Hint*:Make the change of variables $u(x,t) = e^{-bt}v(x,t)$.)

18. Solve the heat equation with convection:
$$u_t - ku_{xx} + Vu_x = 0 \qquad \text{for} - \infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$
where V is a constant.(*Hint*:Go to a moving frame of reference by substituting $y = x - Vt$.)

Exercise 2.5

1. Show that there is no maximum principle for the wave equation.

Suggested Solution to Assignment 2

Exercise 2.1

1. By d'Alembert's formula, the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[e^{x+ct}+e^{x-ct}] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s ds \\ &= \frac{1}{2}[e^{x+ct}+e^{x-ct}] + \frac{1}{2c}[\cos(x-ct)-\cos(x+ct)]. \quad \square \end{aligned}$$

2. By d'Alembert's formula, the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\{\log[1+(x+ct)^2]+\log[1+(x-ct)^2]\} + \frac{1}{2c}\int_{x-ct}^{x+ct} (4+s) ds \\ &= \frac{1}{2}\{\log[1+(x+ct)^2]+\log[1+(x-ct)^2]\} + 4t + xt. \quad \square \end{aligned}$$

5. By d'Alembert's formula, the solution is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s) ds = \frac{1}{2c}[\text{length of } (x-ct, x+ct)\cap(-a,a)].$$

So we have

$$u(x,a/2c) = \begin{cases} 0 & x\in(-\infty,-\frac{3a}{2}]\cup[\frac{3a}{2},\infty); \\ \frac{1}{2c}(\frac{3a}{2}-x) & x\in[\frac{a}{2},\frac{3a}{2}]; \\ \frac{a}{2c} & x\in[-\frac{a}{2},\frac{a}{2}]; \\ \frac{1}{2c}(\frac{3a}{2}+x) & x\in[-\frac{3a}{2},-\frac{a}{2}]; \end{cases}
\qquad u(x,a/c) = \begin{cases} 0 & x\in(-\infty,-2a]\cup[2a,\infty); \\ \frac{1}{2c}(2a-x) & x\in[0,2a]; \\ \frac{1}{2c}(2a+x) & x\in[-2a,0]; \end{cases}$$

$$u(x,3a/2c) = \begin{cases} 0 & x\in(-\infty,-\frac{5a}{2}]\cup[\frac{5a}{2},\infty); \\ \frac{1}{2c}(\frac{5a}{2}-x) & x\in[\frac{a}{2},\frac{5a}{2}]; \\ \frac{a}{c} & x\in[-\frac{a}{2},\frac{a}{2}]; \\ \frac{1}{2c}(\frac{5a}{2}+x) & x\in[-\frac{5a}{2},-\frac{a}{2}]; \end{cases}
\qquad u(x,2a/c) = \begin{cases} 0 & x\in(-\infty,-3a]\cup[3a,\infty); \\ \frac{1}{2c}(3a-x) & x\in[a,3a]; \\ \frac{a}{c} & x\in[-a,a]; \\ \frac{1}{2c}(3a+x) & x\in[-3a,-a]; \end{cases}$$

$$u(x,5a/c) = \begin{cases} 0 & x\in(-\infty,-6a]\cup[6a,\infty); \\ \frac{1}{2c}(6a-x) & x\in[4a,6a]; \\ \frac{a}{c} & x\in[-4a,4a]; \\ \frac{1}{2c}(6a+x) & x\in[-6a,-4a]; \end{cases}$$

Here we omit the figures. $\square$

6.

$$\max_x u(x,t) = \begin{cases} t & 0\le t\le \frac{a}{c}; \\ \frac{a}{c} & t\ge \frac{a}{c}. \end{cases} \qquad \square$$

7. Since ϕ and ψ are odd function of x,

$$\begin{aligned} u(-x,t) &= \frac{1}{2}[\phi(-x+ct)+\phi(-x-ct)] + \frac{1}{2c}\int_{-x-ct}^{-x+ct}\psi(s)ds \\ &= \frac{1}{2}[-\phi(x-ct)-\phi(x+ct)] + \frac{1}{2c}\int_{x+ct}^{x-ct}\psi(-s)d(-s) \\ &= -\{\frac{1}{2}[\phi(x-ct)+\phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)d(s)\} = -u(x,t). \end{aligned}$$

Thus $u(x,t)$ is odd in x for all t. □

8. (a) Change variables $v = ru$, then

$$v_{tt} = ru_{tt}, v_{rr} = (ru_r + u)_r = ru_{rr} + 2u_r,$$

which implies

$$v_{tt} = rc^2(u_{rr} + \frac{2}{r}u_r) = c^2 v_{rr}$$

(b) Using the same skill related to the wave equation(1), we have $v(r,t) = f(r+ct) + g(r-ct)$, where f and g are two arbitrary functions of a single variable. Hence $u = \frac{1}{r}f(r+ct) + \frac{1}{r}g(r-ct)$.

(c) Since $v(r,0) = r\phi(r)$ and $v_t(r,0) = r\psi(r)$ are both odd, we can extend v to all of $\mathbb{R}$ by odd reflection. That is, we set

$$\tilde{v}(r,t) = \begin{cases} v(r,t), & r > 0; \\ 0, & r = 0; \\ -v(-r,t), & r < 0. \end{cases}$$

Hence d'Alembert's formula implies

$$\tilde{v}(r,t) = \frac{1}{2}[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] - \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)ds.$$

Therefore for $r > 0$,

$$u(r,t) = \frac{1}{r}v(r,t) = \frac{1}{2r}[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] - \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)ds. \quad \square$$

10. Using the same way above, since $(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t})(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial t})u = 0$, we can obtain that the general solution is $u(x,t) = f(x + \frac{1}{4}t) + g(x - \frac{1}{5}t)$. The initial conditions implies

$$f(x) = \frac{1}{9}[4\phi(x) + 20\int_0^x \psi(s)ds + C],\ g(x) = \frac{1}{9}[5\phi(x) - 20\int_0^x \psi(s)ds - C].$$

Therefore, the solution is

$$u(x,t) = \frac{1}{9}[4\phi(x + \frac{1}{4}t) + 5\phi(x - \frac{1}{5}t)] + \frac{20}{9}\int_{x-\frac{1}{5}t}^{x+\frac{1}{4}t}\psi(s)ds. \quad \square$$

Exercise 2.2

1. By the law of conservation of energy, $E = \frac{1}{2}\int_{-\infty}^{\infty}(\rho u_t^2 + T u_x^2)\ dx$ is a constant independent of t. Since $\phi \equiv 0$ and $\psi \equiv 0$, we have $E \equiv 0$. Thus, the first vanishing theorem implies $u_t \equiv 0$ and $u_x \equiv 0$. So $u \equiv 0$ since $\phi \equiv 0$. $\square$

2. (a) By the chain rule,

$$\partial e/\partial t = u_t u_{tt} + u_x u_{xt},\ \partial e/\partial x = u_t u_{tx} + u_x u_{xx},$$
$$\partial p/\partial t = u_t u_{xt} + u_{tt} u_x,\ \partial p/\partial x = u_t u_{xx} + u_{tx} u_x.$$

Since $u_{tt} = u_{xx}$ and $u_{xt} = u_{tx}$,

$$\partial e/\partial t = \partial p/\partial x,\ \ \partial e/\partial x = \partial p/\partial t.$$

(b) From the result of (a),

$$e_{tt} = p_{xt} = p_{tx} = e_{xx},\ p_{tt} = e_{xt} = e_{tx} = p_{xx}.$$

So both $e(x,t)$ and $p(x,t)$ satisfy the wave equation. $\square$

3. (a) $(u(x-y,t))_{tt} = u_{tt}(x-y,t) = c^2 u_{xx}(x-y,t) = c^2\left(u(x-y,t)\right)_{xx}$.

(b) $(u_x(x,t))_{tt} = u_{xtt}(x,t) = c^2 u_{xxx}(x,t) = c^2\left(u_x(x,t)\right)_{xx}$.

(c) $(u(ax,at))_{tt} = a^2 u_{tt}(ax,at) = a^2c^2 u_{xx}(ax,at) = c^2\left(u(ax,at)\right)_{xx}$. $\square$

5. For damped string, $u_{tt} - c^2 u_{xx} + r u_t = 0$, where $c = \sqrt{\frac{T}{\rho}}$, the energy is

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \rho(u_t^2 + c^2 u_x^2) dx.$$

Hence,

$$\begin{aligned} dE/dt &= \frac{1}{2}\int_{-\infty}^{\infty} \rho(2u_t u_{tt} + 2c^2 u_x u_{xt}) dx \\ &= \int_{-\infty}^{\infty} \rho(c^2 u_t u_{xx} - r u_t^2 + c^2 u_x u_{xt}) dx \\ &= \int_{-\infty}^{\infty} \rho(c^2 u_t u_{xx} - r u_t^2 - c^2 u_{xx} u_t) dx + (c^2 u_t u_x)\Big|_{-\infty}^{\infty} \\ &= -\int_{-\infty}^{\infty} \rho r u_t^2 dx \le 0. \quad \square \end{aligned}$$

Exercise 2.3

2. By the definition of maximum and minimum, $M(T)$ increases(i.e. nondecreasing) and $m(T)$ decreases(i.e. nonincreasing). $\square$

3. (a) Use the strong minimum principle, we omit the details here.

(b) Use the minimum principle. Since $u(0,t) = u(1,t) = 0$, $u(x,t) \ge u(x,t_0)$ for $\forall t_0 \le t < 1$. So $\mu(t)$ is dereasing.
Or let the maximum occur at point $X(t)$, so that $\mu(t) = u(X(t),t)$. Differentiale $\mu(t)$, assuming that $X(t)$ is differentiable, we have

$$\mu'(t) = u_x(X(t),t)X'(t) + u_t(X(t),t)$$

Note at point $(X(t),t)$ we have $u_x = 0, u_{xx} \le 0$. Hence, $\mu'(t) = u_{xx}(X(t),t) \le 0$ and $\mu(t)$ is decreasing.

(c) Here we omit the figure. Note that $u(0,t) = u(1,t) = 0$ and the result in (b). $\square$

4. (a) Note that $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x) \in [0,1]$. Then the conclusion can be verified by strong maximum principle.

   (b) Let $v(x,t) = u(1-x,t)$, then $v(0,t) = v(1,t) = 0$ and $v(x,0) = 4x(1-x) = u(x,0)$. Then the uniqueness theorem for the diffusion theorem implies $u(x,t) = u(1-x,t)$.

   (c)
$$\frac{d}{dt}\int_0^1 u^2 dx = \int_0^1 2uu_t dx = 2\int_0^1 uu_{xx} dx = -2\int_0^1 u_x^2 dx.$$
   Since $u(x,t) > 0$ for all $t > 0$ and $0 < x < 1$, so u_x is not zero function. Hence, $\frac{d}{dt}\int_0^1 u^2 dx < 0$ and $\int_0^1 u^2 dx$ is a strictly decreasing function of t. $\square$

5. (a) We omit the details to verify that $u = -2xt - x^2$ is a solution. When t is fixed, u attains its maximum at $(-t,t)$ and $u(-t,t) = t^2$. So u attains its maximum at $(-1,1)$ in the closed rectangle $\{-2 \le x \le 2, 0 \le t \le 1\}$.

   (b) In our proof the maximum principle for the diffusion equation, the key point is that $v(x,t) = u(x,t) + \epsilon x^2$ satisfies $v_t - kv_{xx} < 0$. However, here $v_t - kv_{xx} = u_t - x(u+\epsilon x^2)_{xx} = -2\epsilon x$ so that the sign of $v_t - kv_{xx}$ is not unchanged in the closed rectangle $\{-2 \le x \le 2, 0 \le t \le 1\}$. $\square$

6. Let $w = u - v$ and use maximum principle for the diffusion equation. We omit the details. $\square$

7. (a) Let $w(x,t) = u(x,t) - v(x,t)$ and $w_\epsilon(x,t) = w(x,t) + \epsilon x^2$. Since $w_t - kw_{xx} = f - g \le 0$, we can use the same method in the text book to derive the maximum principle for w. So $u \le v$ at $x = 0$, $x = l$ and $t = 0$ implies $w \le 0$ in the rectangle, i.e. $u \le v$ for $0 \le x \le l$, $0 \le t < \infty$. Here we omit the details of the method in the text book.

   (b) Let $u(x,t) = (1-e^{-t})\sin x$, and then $u_t - u_{xx} = \sin x$ and $u = 0$ at $x = 0$, $x = \pi$ and $t = 0$. Therefore, the result above implies $v(x,t) \ge (1-e^{-t})\sin x$. $\square$.

Extra 1. (1) Define $v(x,t) := e^{-at}u(x,t)$, then $v_t = kv_{xx}, V(0,t) = v(1,t) = 0, v(x,0) = sin(\pi x)$. By the Strong Maximum Principle, $0 < v(x,t) < 1, \forall t > 0, 0 < x < 1$. Thus, $0 < u(x,t) = e^{at}v(x,t) < 1, \forall t > 0, 0 < x < 1$

(2)Define $v(x,t) := u(1-x,t)$, then we can easily check that v solves the same problem as u. By the uniqueness of the solution, $u = v$

Extra 2. (a)Follow the proof of the Maximum Principle in the textbook. We only need to change the diffusion inequality (2) in Page 42 to be

$$v_t - kv_{xx} = u_t - ku_{xx} - 2\varepsilon k \le -2\varepsilon k < 0$$

(b)Define $u(x,t) := v(x,t) - t\max_{-\infty<x<+\infty,0<t<T} f(x,t)$, then

$$u_t - ku_{xx} = v_t - \max_{-\infty<x<+\infty,0<t<T} f(x,t) - kv_{xx} = f - \max_{-\infty<x<+\infty,0<t<T} f(x,t) \le 0$$

$$\Rightarrow \max_{-\infty<x<+\infty,0\le t\le T} u(x,t) = \max_{-\infty<x<+\infty,t=0} u(x,t) = 0, by(a)$$

$$\Rightarrow v(x,t) \le t\max_{-\infty<x<+\infty,0<t<T} f(x,t) \le T\max_{-\infty<x<+\infty,0<t<T} f(x,t)$$

Exercise 2.4

1. By the general formula,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_{-l}^{l} e^{-(x-y)^2/4kt}dy \\ &= \frac{1}{\sqrt{\pi}}\int_{(-l-x)/\sqrt{4kt}}^{(l-x)/\sqrt{4kt}} e^{-p^2}dp \\ &= \frac{1}{2}\{\mathscr{E}rf[\frac{x+l}{\sqrt{4kt}}] - \mathscr{E}rf[\frac{x-l}{\sqrt{4kt}}]\}. \quad \square \end{aligned}$$

2. By the general formula,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty} e^{-(x-y)^2/4kt}dy + \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{0} 3e^{-(x-y)^2/4kt}dy \\ &= \frac{1}{2} + \frac{1}{2}\mathscr{E}rf[\frac{x}{\sqrt{4kt}}] + \frac{3}{2} - \frac{3}{2}\mathscr{E}rf[\frac{x}{\sqrt{4kt}}] \\ &= 2 - \mathscr{E}rf[\frac{x}{\sqrt{4kt}}]. \quad \square \end{aligned}$$

5. Similar to Exercise 2.2.3.

8. By the definition of $S(x,t)$,

$$\max_{\delta \le x < \infty} = \frac{1}{\sqrt{4\pi kt}}e^{-\delta^2/4kt},$$

so

$$\lim_{t\to 0^+}\max_{\delta\le x<\infty} = \lim_{t\to 0^+}\frac{1}{\sqrt{4\pi kt}}e^{-\delta^2/4kt} = \lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{4\pi k}}e^{-x\delta^2/4k} = 0. \quad \square$$

11. (a) Since $u(x,t)$ and $-u(-x,t)$ are the solutions and $u(x,0) = \phi(x) = -\phi(-x) = -u(-x,0)$, it follows from the uniqueness theorem that $u(x,t) = -u(-x,t)$.

    (b) Similar to (a).

    (c) Similar to (a). $\square$

14. Since

$$|e^{-(x-y)^2/4kt}\phi(y)| \le Ce^{-(x-y)^2/4kt+ay^2} = Ce^{(a-\frac{1}{4kt})y^2+\frac{x}{2kt}y-\frac{x^2}{4kt}},$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy$$

makes sense for $a - \frac{1}{4kt} < 0$, i.e. $0 < t < 1/(4ak)$, but not necessarily for large t, for example, $\phi(x) = e^{ax^2}$. $\square$

15. Suppose that both u and v are solution of the diffusion problem with the same Neumann boundary condition. Let $w(x,t) = u(x,t) - v(x,t)$, then w satisfies

$$w_t = kw_{xx}, \quad w(x,0) = w_x(0,t) = w_x(l,t) = 0.$$

Thus by the integration by part and the Neumann boundary condition,

$$\frac{d}{dt}\int_0^l \frac{1}{2}w^2(x,t)dx = -k\int_0^l w_x^2(x,t)dx \le 0.$$

Hence, the initial condition implies

$$\int_0^l \frac{1}{2}w^2(x,t)dx \le \int_0^l \frac{1}{2}w^2(x,0)dx = 0.$$

Therefor, $w = 0$, i.e. $u = v$ for all $t > 0$. $\square$

16. Let $v(x,t) = e^{bt}u(x,t)$, then v satisfies

$$v_t - kv_{xx} = 0, \quad v(x,0) = u(x,0) = \phi(x).$$

Hence, the general solution of v is

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy,$$

and the general solution of u is

$$u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy. \quad \square$$

18. Let $v(x,t) = u(x+Vt,t)$, then v satisfies

$$v_t - kv_{xx} = 0, \quad v(x,0) = u(x,0) = \phi(x).$$

Since

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy,$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-Vt-y)^2/4kt}\phi(y)\ dy. \quad \square$$

Exercise 2.5

1. Let $u(x,t) = -x^2 - (t-1)^2$ be the unique solution of the wave equation with boundary conditions:

$$u_{tt} = u_{xx}, \text{ for } -1 < x < 1, 0 < t < \infty,$$

$$u(x,0) = -x^2 - 1, \ u_t(x,0) = 2,$$

$$u(-1,t) = u(1,t) = -t^2 + 2t - 2.$$

But u attains its maximum 0 at $(0,1)$. $\square$